iscc Tutorial

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Outline

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Introduction

- What is `iscc`?
  - ⇒ interactive interface to the `barvinok` counting library
  - ⇒ also provides interface to the `CLooG` code generation library and to some operations of the `isl` integer set library
  - ⇒ inspired by Omega Calculator from the Omega Project

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  - ⇒ available from `http://freshmeat.net/projects/barvinok/`

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- How to run `iscc`?
  - ⇒ optionally obtain `CLooG` from `http://www.cloog.org/`
  - ⇒ compile and install `barvinok` following the instructions in `README`
  - ⇒ run `iscc`
    Note: `iscc` currently does not use readline, so you may want to use a readline front-end: `rlwrap iscc`

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Examples from polyhedral model for program analysis and transformation

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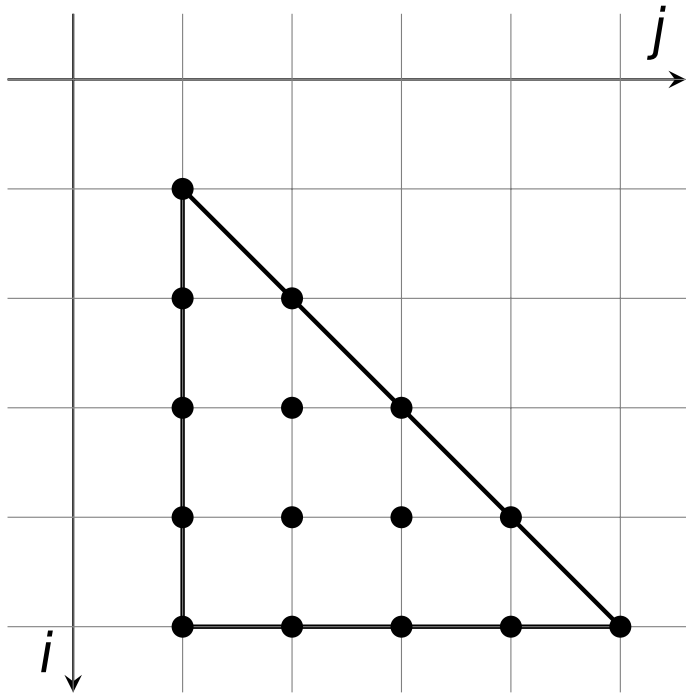

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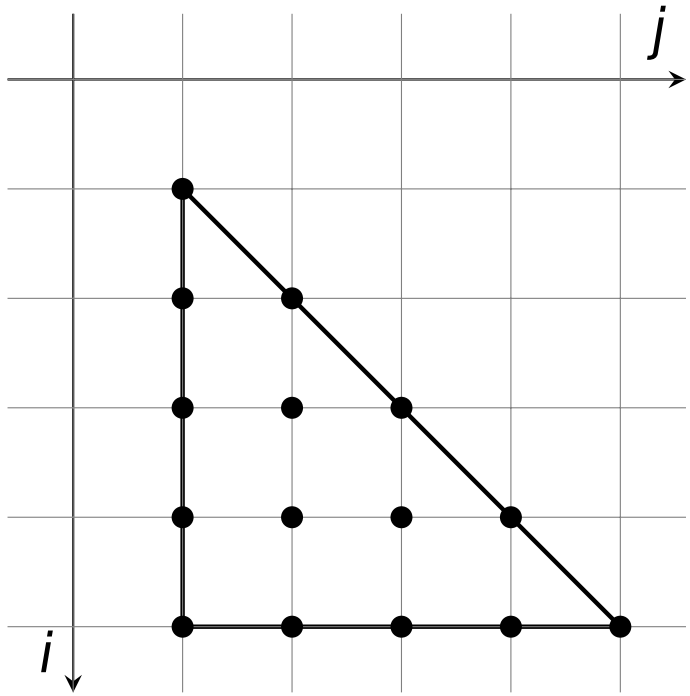


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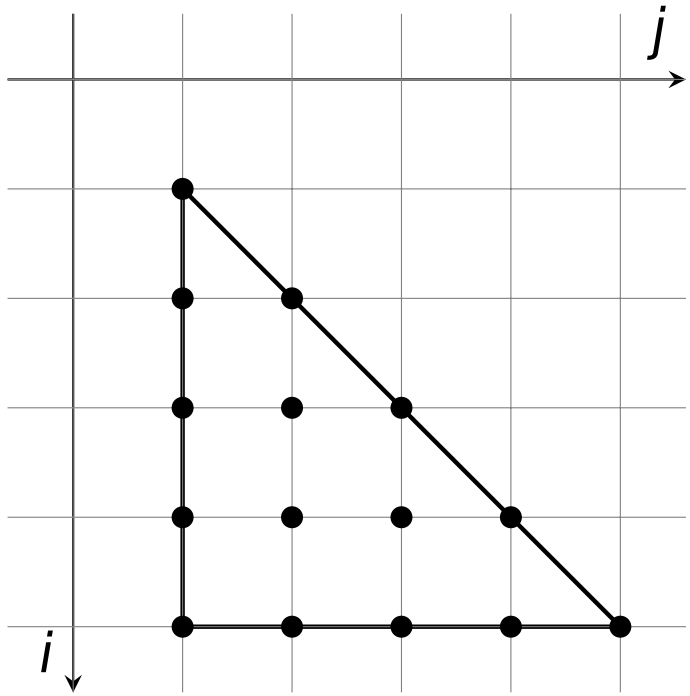

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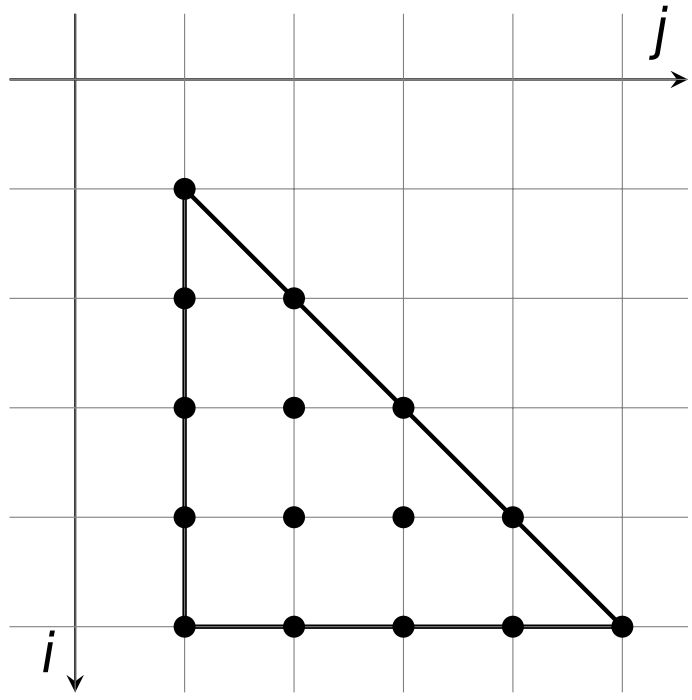

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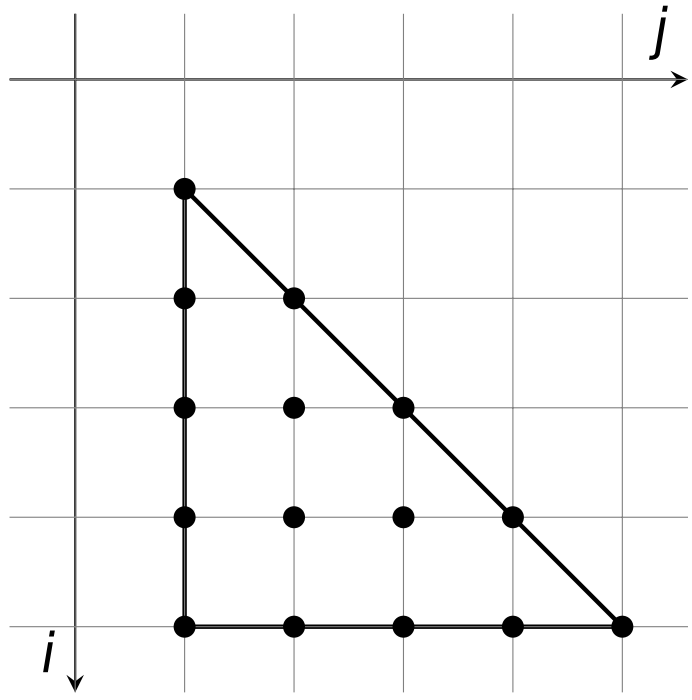

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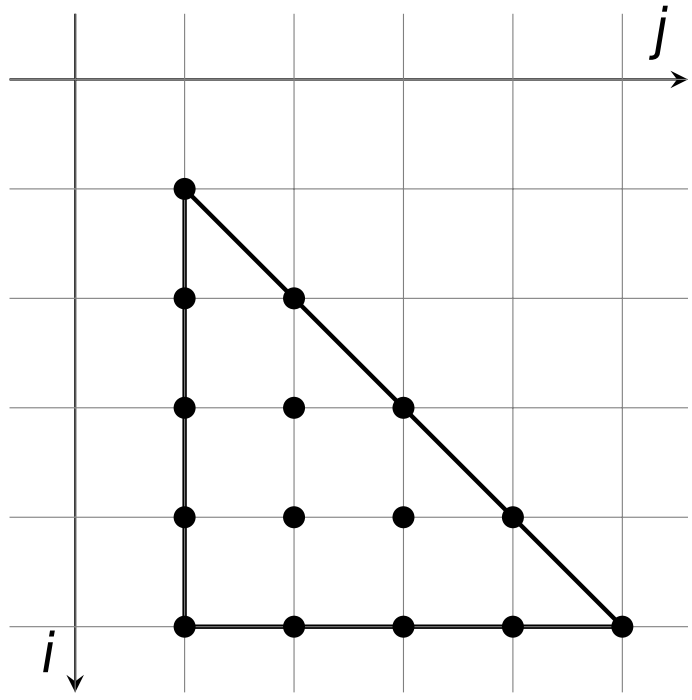

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parameters

affine constraints in DNF

Set Variables and Parameters

- set variables
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  - identified by position
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```
[n] -> { [i,j] : 1 <= i <= n and 1 <= j <= i }
```

is equal to

```
[n] -> { [a,b] : 1 <= a <= n and 1 <= b <= a }
```

but not equal to

```
[n] -> { [j,i] : 1 <= i <= n and 1 <= j <= i }
```

or

```
[m] -> { [i,j] : 1 <= i <= m and 1 <= j <= i }
```

Code Generation

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for (i = 1; i <= n; ++i)
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codegen [n] -> { [i,j] : 1 <= i <= n and 1 <= j <= i };
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⇒ generate code that visits elements in lexicographic order

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What if a different order is needed?
⇒ apply a schedule: maps iterations domain to multi-dimensional time
⇒ multi-dimensional time is ordered lexicographically

Example: interchange `i` and `j`
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intersect domain of map on the left with set on the right

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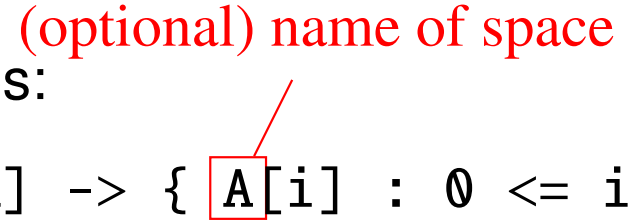

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(optional) name of space

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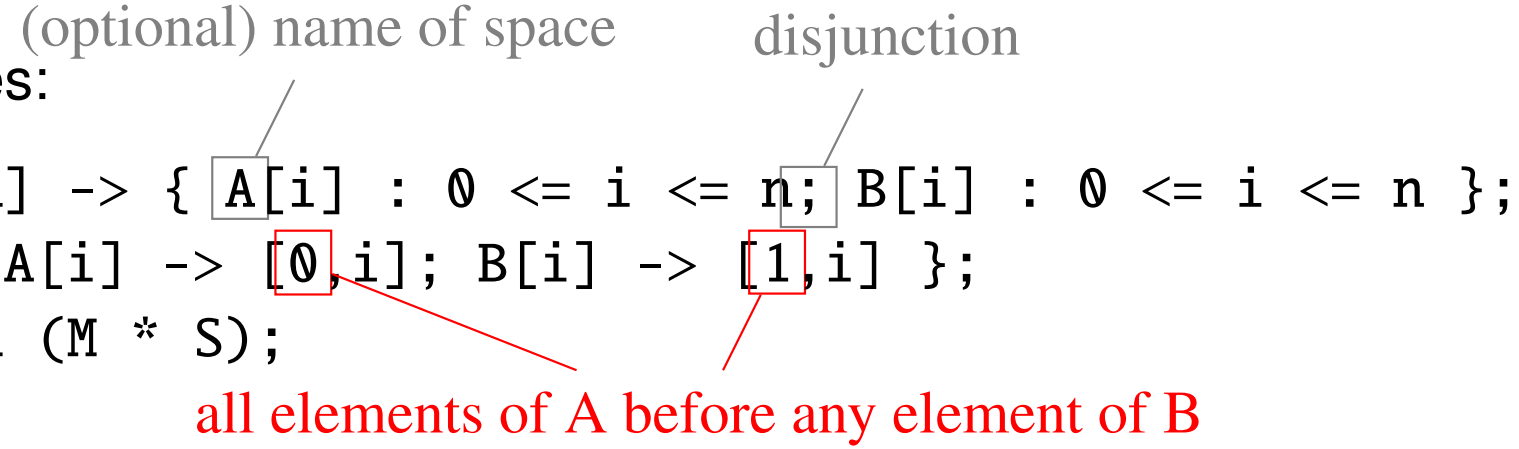

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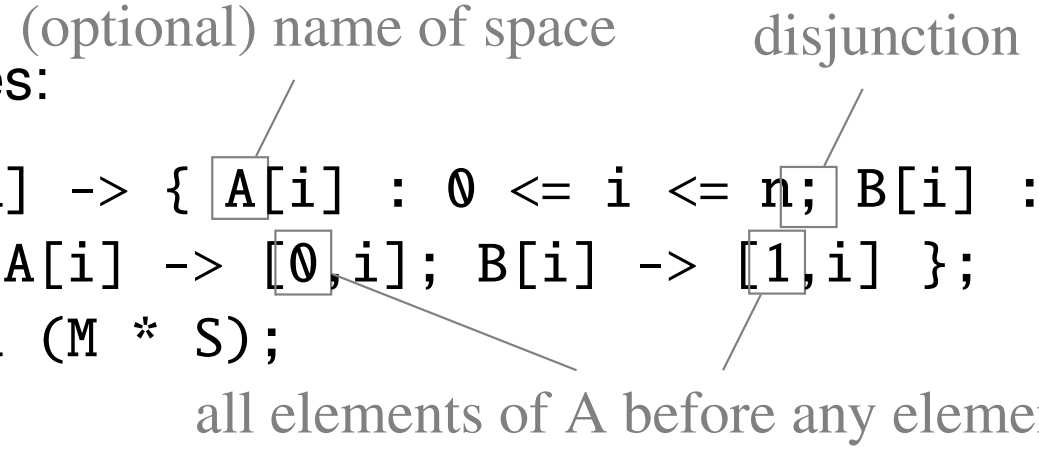


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Lexicographic Optimization

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for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
        a[i+j] = f(a[i+j]);
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- What is the last iteration of the loop?

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Dependence Analysis

Given a read from an array element, what was the last write to the same array element before the read?

Simple case: array written through a single access

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In general: impose lexicographical order on shared iterators

Transitive Closures

Given a graph (represented as an affine map)

```
M := { A[i] -> A[i+1] : 0 <= i <= 3; B[] -> A[2] };
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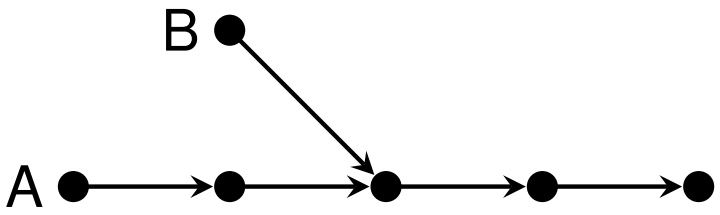

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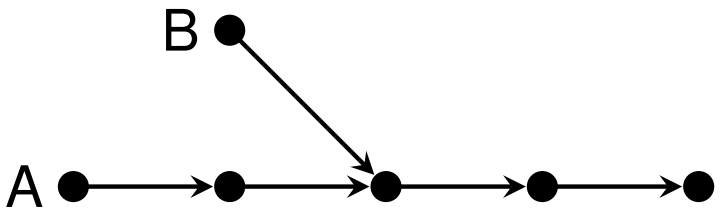

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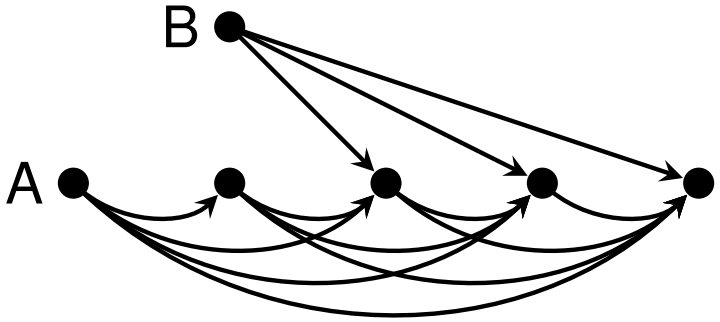

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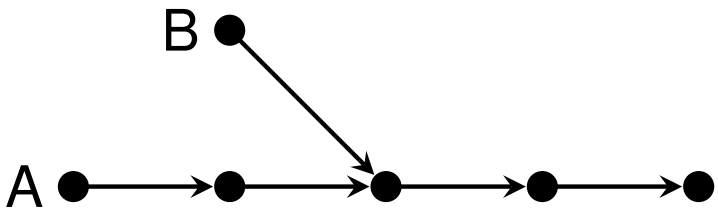

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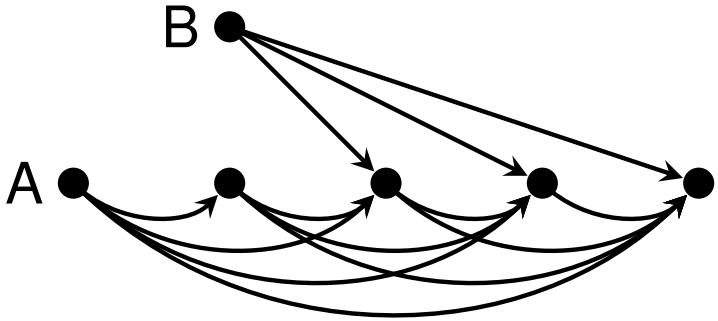

Result:

```
({ B[] -> A[o0] : o0 <= 4 and o0 >= 3; B[] -> A[2];
   A[i] -> A[o0] : i >= 0 and i <= 3 and o0 >= 1 and
   o0 <= 4 and o0 >= 1 + i }, True)
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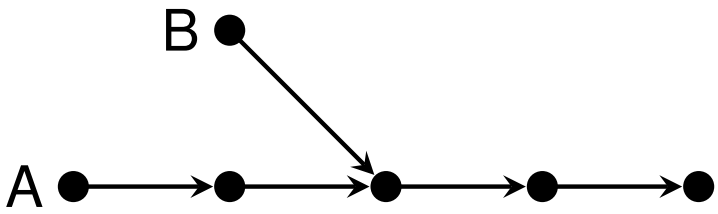

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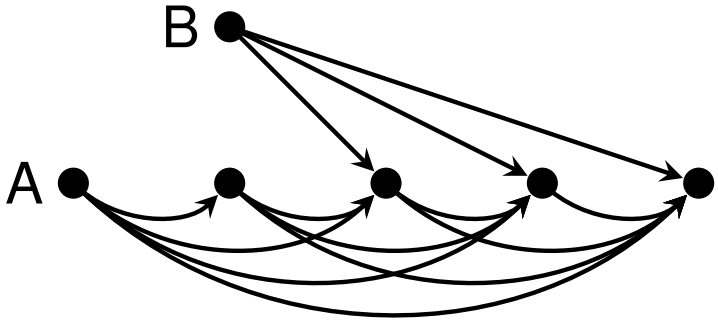

Result:

exact transitive closure

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Cardinality

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for (i = 0; i < N; ++i)
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- How many times is the statement executed?

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card S;
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- How many times is a given array element written?

```
A:=[N]->{[i,j]->a[i+j]:0<=i<N and 0<=j<N-i};
card (A^-1);
```

Cardinality

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
        a[i+j] = f(a[i+j]);
```

- How many times is the statement executed?

```
S := [N] -> { [i,j] : 0<=i<N and 0<=j<N-i };
card S;
```

number of elements in the set

- How many times is a given array element written?

```
A:=[N]->{[i,j]->a[i+j]:0<=i<N and 0<=j<N-i};
card (A^-1);
```

number of image elements

Cardinality

```
for (i = 0; i < N; ++i)
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        a[i+j] = f(a[i+j]);
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A:=[N]->{[i,j]->a[i+j]:0<=i<N and 0<=j<N-i};
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```

number of image elements

- How many array elements are written?

```
A:=[N]->{[i,j]->a[i+j]:0<=i<N and 0<=j<N-i};
card (ran A);
```

Quasipolynomials

```
for (i = 1; i <= n; ++i)
    for (j = 1; j <= n - 2 * i; ++j)
        /* S */
```

How many times is S executed?

```
card [n] -> { [i,j] : 1 <= i <= n and 1 <= j <= n - 2i };
```

Quasipolynomials

```
for (i = 1; i <= n; ++i)
    for (j = 1; j <= n - 2 * i; ++j)
        /* S */
```

How many times is S executed?

```
card [n] -> { [i,j] : 1 <= i <= n and 1 <= j <= n - 2i };
```

Result:

```
[n] -> { ((-1/4 * n + 1/4 * n^2) - 1/2 * [(n)/2]) : n >= 3 }
```

That is,

$$-\frac{n}{4} + \frac{n^2}{4} - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \quad \text{if } n \geq 3.$$

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for (i = 1; i <= n; ++i)
    for (j = 1; j <= n - 2 * i; ++j)
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How many times is S executed?

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card [n] -> { [i,j] : 1 <= i <= n and 1 <= j <= n - 2i };
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Result:

greatest integer part

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Polynomial approximations

$\Rightarrow$ run `iscc --polynomial-approximation`

Incremental Counting

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
        a[i+j] = f(a[i+j]);
```

How many times is the statement executed?

- direct computation

```
card [N] -> { [i,j] : 0<=i<N and 0<=j<N-i };
```

Incremental Counting

```
for (i = 0; i < N; ++i)
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How many times is the statement executed?

- direct computation

  ```
  card [N] -> { [i,j] : 0<=i<N and 0<=j<N-i };
  ```

- incremental computation

  ```
  card [N] -> { [i] -> [j] : 0<=i<N and 0<=j<N-i };
  ```

Incremental Counting

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for (i = 0; i < N; ++i)
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  ```
  card [N] -> { [i,j] : 0<=i<N and 0<=j<N-i };
  ```

- incremental computation

  ```
  card [N] -> { [i] -> [j] : 0<=i<N and 0<=j<N-i };
  ```

  Result:

  ```
  [N] -> { [i] -> (N - i) : i <= -1 + N and i >= 0 }
  ```

  ```
  sum [N] -> { [i] -> (N - i) : i <= -1 + N and i >= 0 };
  ```

Incremental Counting

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
        a[i+j] = f(a[i+j]);
```

How many times is the statement executed?

- direct computation

  ```
  card [N] -> { [i,j] : 0<=i<N and 0<=j<N-i };
  ```

- incremental computation

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  Result:

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  ```

  ```
  sum [N] -> { [i] -> (N - i) : i <= -1 + N and i >= 0 };
  ```

  sum over all elements in domain

Total Memory Allocation

```
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        p[i][j] = malloc(i * j + i - N + 1);
/* ... */
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        free(p[i][j]);
```

How much memory allocated in total?

Total Memory Allocation

```
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        p[i][j] = malloc(i * j + i - N + 1);
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How much memory allocated in total?

```
sum [N] -> {[i,j] -> i*j+i-N+1: 0 <= i < N and i <= j < N};
```

Memory Requirements

```
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j) {
        p = malloc(i * j + i - N + 1);
        /* ... */
        free(p);
    }
```

How much memory is needed?

Memory Requirements

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for (i = 0; i < N; ++i)
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ub [N] -> {[i,j] -> i*j+i-N+1: 0 <= i < N and i <= j < N};
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Result:

```
([N] -> { max((1 - 2 * N + N^2)) : N >= 1 }, True)
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```

Result:

```
([N] -> { max((1 - 2 * N + N^2)) : N >= 1 }, True)
```

bound is tight

Outline

Pointer Conversion

```
p = a;
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j) {
        p += j * ((j-i)/4);
        *p = hard_work(i,j);
    }
```

Can we parallelize this code?

Pointer Conversion

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p = a;
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j) {
        p += j * ((j-i)/4);
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    }
```

Can we parallelize this code?

⇒ No, (false) dependency through `p`
⇒ Compute closed formula for `p`

$$p = a + \sum_{\substack{(i',j') \in S \\ (i',j') \preccurlyeq (i,j)}} j' \left\lfloor \frac{j' - i'}{4} \right\rfloor$$

with $S = \{ (i', j') \in \mathbb{Z}^2 \mid 0 \leq i' < N \wedge i' \leq j' < N \}$

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$\preccurlyeq$ lexicographically less than

with $S = \{ (i', j') \in \mathbb{Z}^2 \mid 0 \leq i' < N \wedge i' \leq j' < N \}$

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```
S := [N] -> { [i,j] : 0 <= i < N and i <= j < N };
L := S <<= S;
INC := { [[i,j] -> [i',j']] -> j' * [(j'-i')/4] };
INC := INC * (wrap (L^-1));
sum INC;
```

Pointer Conversion

$$p = a + \sum_{\substack{(i',j') \in S \\ (i',j') \preccurlyeq (i,j)}} j' \left\lfloor \frac{j' - i'}{4} \right\rfloor$$

with $S = \{ (i', j') \in \mathbb{Z}^2 \mid 0 \leq i' < N \wedge i' \leq j' < N \}$

map: (elements of) left set lexicographically smaller than right set

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S := [N] -> { [i,j] : 0 <= i < N and i <= j < N };
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embed map in a set

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INC := INC * (wrap (L^-1));
sum INC;
```

embed map in a set

Note: if domain of argument to `sum` [ub] is an embedded map, then sum [bound] is computed over range of embedded map

Dynamic Memory Requirement Estimation [CFGV2006]

How much memory is needed to execute the following program?

```
void m0(int m) {
    for (c = 0; c < m; c++) {
        m1(c);
        B[] m2Arr = m2(2 * m - c);
    }
}
void m1(int k) {
    for (i = 1; i <= k; i++) {
        A a = new A();
        B[] dummyArr = m2(i);
    }
}
B[] m2(int n) {
    B[] arrB = new B[n];
    for (j = 1; j <= n; j++)
        B b = new B();
    return arrB;
}
```

Dynamic Memory Requirement Estimation [CFGV2006]

How much (scoped) memory is needed?
$\Rightarrow$ compute for each method

$\mathrm{ret}_{\mathtt{m}}$ size of memory returned by $\mathtt{m}$

$\mathrm{cap}_{\mathtt{m}}$ size of memory "captured" (not returned) by $\mathtt{m}$

$\mathrm{memRq}_{\mathtt{m}}$ total memory requirements of $\mathtt{m}$

$$\mathrm{memRq}_{\mathtt{m}} = \mathrm{cap}_{\mathtt{m}} + \max_{\mathtt{p}\ \text{called by}\ \mathtt{m}} \mathrm{memRq}_{\mathtt{p}}$$

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```
B[] m2(int n) {
 B[] arrB = new B[n];
 for (j=1; j<=n; j++)
  B b = new B();
 return arrB;
}
```

Dynamic Memory Requirement Estimation [CFGV2006]

How much (scoped) memory is needed?
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ret_m size of memory returned by m

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$$\text{memRq}_m = \text{cap}_m + \max_{p \text{ called by } m} \text{memRq}_p$$

```
B[] m2(int n) {
 B[] arrB = new B[n];
 for (j=1; j<=n; j++)
  B b = new B();
 return arrB;
}
```

```
ret_m2 := { [n] -> n : n >= 0 };
cap_m2 := sum { [[n]->[j]]->1 :
                  1 <= j <= n };
memrq_m2 := cap_m2;
```

Dynamic Memory Requirement Estimation [CFGV2006]

```
void m1(int k) {
    for (i = 1; i <= k; i++) {
        A a = new A();
        B[] dummyArr = m2(i);
    }
}
```

$$\mathrm{cap}_{\mathtt{m1}}(k) = \sum_{1 \le i \le k} (1 + \mathrm{ret}_{\mathtt{m2}}(i))$$

`ret_m2` is a function of the parameters of `m2`

We want to use it as a function of the parameters and local variables of `m1`

Dynamic Memory Requirement Estimation [CFGV2006]

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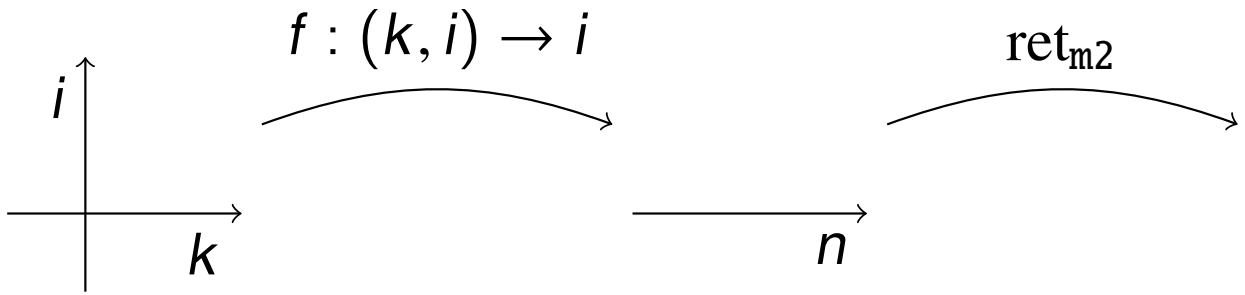

$\Rightarrow$ compose f with $\mathrm{ret}_{\mathtt{m2}}$ (pull back $\mathrm{ret}_{\mathtt{m2}}$ over f)

```
{[[k]->[i]]->[i] : 1 <= i <= k} . ret_m2;
```

Compositions with Piecewise (Reductions of) Quasipolynomials

`f . g;`

- `f` is a map
- `g` may be
  - piecewise quasipolynomial
    (result of counting problems)

    ⇒ take sum over intersection of `ran f` and `dom g`
  - piecewise reduction of quasipolynomials
    (result of upper bound computation)

    ⇒ compute bound over intersection of `ran f` and `dom g`

Note: if *f* is single-valued, then sum/bound is computed over a single point

Dynamic Memory Requirement Estimation [CFGV2006]

```
void m1(int k) {
    for (i = 1; i <= k; i++) {
        A a = new A();
        B[] dummyArr = m2(i);
    }
}
```

$$\mathrm{memRq_m} = \mathrm{cap_m} + \max_{\text{p called by m}} \mathrm{memRq_p}$$

```
ret_m1 := { [k] -> 0 };
alloc1 := { [[k] -> [i]] -> 1 : 1 <= i <= k };
alloc2 := {[[k]->[i]]->[i] : 1 <= i <= k} . ret_m2;
cap_m1 := sum (alloc1 + alloc2);
memrq_m1 := cap_m1 +
    (ub ({ [[k] -> [i]] -> [i] : 1 <= i <= k } . memrq_m2));
```

Dynamic Memory Requirement Estimation [CFGV2006]

```
void m0(int m) {
    for (c = 0; c < m; c++) {
        m1(c);
        B[] m2Arr = m2(2 * m - c);
    }
}
```

```
ret_m0 := { [m] -> 0 };
a1 := { [m] -> [c] : 0 <= c < m } . ret_m1;
a2 := {[m]->[m,c] : 0 <= c < m} . {[m,c]->[2m-c]} . ret_m2;
cap_m0 := a1 + a2;
t1 := { [m] -> [c] : 0 <= c < m } . memrq_m1;
t2 := ub ({ [[m]->[c]] -> [2m-c] : 0 <= c < m } . memrq_m2);
memrq_m0 := cap_m0 + (t1 . t2);
```

Dynamic Memory Requirement Estimation [CFGV2006]

```
void m0(int m) {
    for (c = 0; c < m; c++) {
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        B[] m2Arr = m2(2 * m - c);
    }
}
```

```
ret_m0 := { [m] -> 0 };
a1 := { [m] -> [c] : 0 <= c < m } . ret_m1;
a2 := {[m]->[m,c] : 0 <= c < m} . {[m,c]->[2m-c]} . ret_m2;
cap_m0 := a1 + a2;
t1 := { [m] -> [c] : 0 <= c < m } . memrq_m1;
t2 := ub ({ [[m]->[c]] -> [2m-c] : 0 <= c < m } . memrq_m2);
memrq_m0 := cap_m0 + (t1 . t2);
```

combine reductions

Reuse Distance Computation

Given an access to a cache line ℓ, how many distinct cache lines have been accessed since the previous access to ℓ?

$\Rightarrow$ Is the cache line still in the cache?

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```
for (i = 0; i <= 7; ++i) {
    A[i];              //reference a
    A[7-i];            //reference b
    if (i <= 3)
        A[2*i];        //reference c
}
```

Assume `A[i]` in cache line $\lfloor i/3 \rfloor$

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Assume `A[i]` in cache line $\lfloor i/3 \rfloor$

i	0			1			2			3			4		5		6		7	
r	a	b	c	a	b	c	a	b	c	a	b	c	a	b	a	b	a	b	a	b
$r@i$	0	7	0	1	6	2	2	5	4	3	4	6	4	3	5	2	6	1	7	0
$\lfloor (r@i)/3 \rfloor$	0	2	0	0	2	0	0	1	1	1	1	2	1	1	1	0	2	0	2	0
distance	0	0	2	1	2	2	1	0	1	1	1	3	2	1	1	3	3	2	2	2

Reuse Distance Computation

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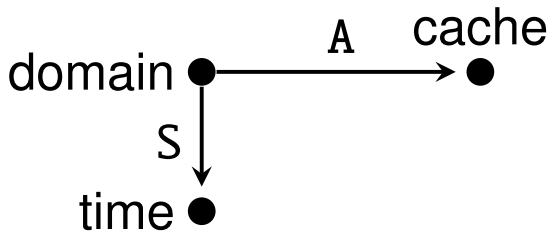

Assume A[i] in cache line $\lfloor i/3 \rfloor$

```
D := { a[i] : 0 <= i <= 7; b[i] : 0 <= i <= 7; c[i] : 0 <= i <= 3 };
C := { A[i] -> L[j] : exists a = [i/3] : j = a };
A := ({ a[i] -> A[i]; b[i] -> A[7-i]; c[i] -> A[2i] } . C) * D;
S := { a[i] -> [i,0]; b[i] -> [i,1]; c[i] -> [i,2] } * D;
```

Reuse Distance Computation

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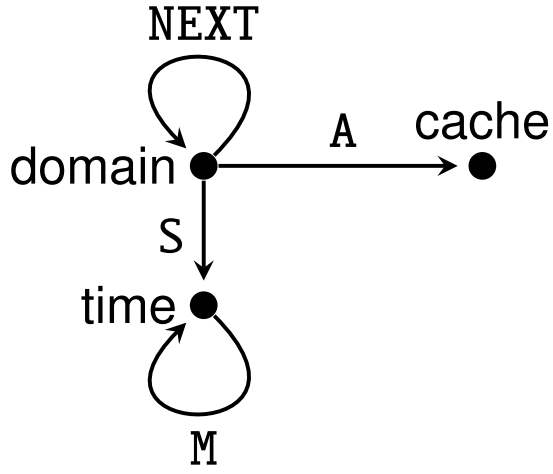

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A := ({ a[i] -> A[i]; b[i] -> A[7-i]; c[i] -> A[2i] } . C) * D;
S := { a[i] -> [i,0]; b[i] -> [i,1]; c[i] -> [i,2] } * D;
TIME := ran S; LT := TIME << TIME; LE := TIME <<= TIME;
T := ((S^-1) . A . (A^-1) . S) * LT;
M := lexmin T;
NEXT := S . M . (S^-1); # map to next access to same cache line
AFTER_PREV := (NEXT^-1) . (S . LE . (S^-1));
BEFORE := S . (LE^-1) . (S^-1);
card ((AFTER_PREV * BEFORE) . A);
```